The Magic Carpet Ride, Hide and Seek

2 You are a young adventurer. Having spent most of your time in the mythical city of Oronto, you decide to leave home for the first time. Your parents want to help you on your journey, so just before your departure, they give you two gifts. Specifically, they give you two forms of transportation: a hover board and a magic carpet. Your parents inform you that both the hover board and the magic carpet have restrictions in how they operate:

We denote the restriction on the hover board's movement by the vector $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$. By this we mean that if the hover board traveled "forward" for one hour, it would move along a "diagonal" path that would result in a displacement of 3 km East and 1 km North of its starting location.

We denote the restriction on the magic carpet's movement by the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. By this we mean that if the magic carpet traveled "forward" for one hour, it would move along a "diagonal" path that would result in a displacement of 1 km East and 2 km North of its starting location.

Scenario Two: Hide-and-Seek

Old Man Gauss wants to move to a cabin in a different location. You are not sure whether Gauss is just trying to test your wits at finding him or if he actually wants to hide somewhere that you can't visit him.

Are there some locations that he can hide and you cannot reach him with these two modes of transportation?

Describe the places that you can reach using a combination of the hover board and the magic carpet and those you cannot. Specify these geometrically and algebraically. Include a symbolic representation using vector notation. Also, include a convincing argument supporting your answer.

Sets and Set Notation

Set

DEFINITION

A ***set*** is a (possibly infinite) collection of items and is notated with curly braces (for example, $\{1,2,3\}$ is the set containing the numbers 1, 2, and 3). We call the items in a set ***elements***.

If X is a set and a is an element of X, we may write $a \in X$, which is read "a is an element of X."

If X is a set, a ***subset*** Y of X (written $Y \subseteq X$) is a set such that every element of Y is an element of X. Two sets are called ***equal*** if they are subsets of each other (i.e., $X = Y$ if $X \subseteq Y$ and $Y \subseteq X$).

We can define a subset using ***set-builder notation***. That is, if X is a set, we can define the subset

$$Y = \{a \in X \ : \ \text{some rule involving } a\},$$

which is read "Y is the set of a in X **such that** some rule involving a is true." If X is intuitive, we may omit it and simply write $Y = \{a : \text{some rule involving } a\}$. You may equivalently use "|" instead of ":", writing $Y = \{a \mid \text{some rule involving } a\}$.

DEFINITION

Some common sets are

$\mathbb{N} = \{\text{natural numbers}\} = \{\text{non-negative whole numbers}\}$.

$\mathbb{Z} = \{\text{integers}\} = \{\text{whole numbers, including negatives}\}$.

$\mathbb{R} = \{\text{real numbers}\}$.

$\mathbb{R}^n = \{\text{vectors in } n\text{-dimensional Euclidean space}\}$.

3

3.1 Which of the following statements are true?

(a) $3 \in \{1,2,3\}$.

(b) $1.5 \in \{1,2,3\}$.

(c) $4 \in \{1,2,3\}$.

(d) "b"$\in \{x \ : \ x \text{ is an English letter}\}$.

(e) "ò"$\in \{x \ : \ x \text{ is an English letter}\}$.

(f) $\{1,2\} \subseteq \{1,2,3\}$.

(g) For some $a \in \{1,2,3\}$, $a \geq 3$.

(h) For any $a \in \{1,2,3\}$, $a \geq 3$.

(i) $1 \subseteq \{1,2,3\}$.

(j) $\{1,2,3\} = \{x \in \mathbb{R} \ : \ 1 \leq x \leq 3\}$.

(k) $\{1,2,3\} = \{x \in \mathbb{Z} \ : \ 1 \leq x \leq 3\}$.

4 Write the following in set-builder notation

4.1 The subset $A \subseteq \mathbb{R}$ of real numbers larger than $\sqrt{2}$.

4.2 The subset $B \subseteq \mathbb{R}^2$ of vectors whose first coordinate is twice the second.

DEFINITION

Unions & Intersections

Let X and Y be sets. The ***union*** of X and Y and the ***intersection*** of X and Y are defined as follows.

$$\text{(union)} \quad X \cup Y = \{a : a \in X \text{ or } a \in Y\}.$$

$$\text{(intersection)} \quad X \cap Y = \{a : a \in X \text{ and } a \in Y\}.$$

5 Let $X = \{1, 2, 3\}$ and $Y = \{2, 3, 4, 5\}$ and $Z = \{4, 5, 6\}$. Compute

5.1 $X \cup Y$

5.2 $X \cap Y$

5.3 $X \cup Y \cup Z$

5.4 $X \cap Y \cap Z$